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Free Large N Supersymmetric Yang-Mills Theory as a String Theory

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Abstract

The strong version of Maldacena's AdS/CFT conjecture implies that the large N expansion of free $\mathcal{N} = 4$ super-YM theory describes an interacting string theory in the extreme limit of high spacetime curvature relative to the string length. String states may then be understood as composed of SYM string bits. We investigate part of the low-lying spectrum of the tensionless (zero-coupling) limit and find a large number of states that are not present in the infinite tension (strong-coupling) limit, notably several massless spin two particles. We observe that all conformal dimensions are N-independent in the free SYM theory, implying that masses in the corresponding string theory are unchanged by string interactions. Degenerate string states do however mix in the interacting string theory because of the complicated N-dependence of general CFT two-point functions. Finally we verify the CFT crossing symmetry, which corresponds to the dual properties of string scattering amplitudes. This means that the SYM operator correlation functions define AdS dual models analogous to the Minkowski dual models that gave rise to string theory.

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1 Introduction

Since 't Hooft's original discussion [1] of the large N behaviour of gauge theories we have had a picture of a topological expansion of gauge theories in terms of surfaces of different genus, resembling the genus expansion of string amplitudes. In recent years Maldacena's conjecture [2], relating the large N expansion of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to string theory in an $AdS_5 \times S^5$ background has stimulated a resurge of interest in the large N limit. The conjectured correspondence permits the calculation of previously inaccessible gauge theory quantities by means of classical supergravity techniques when the 't Hooft coupling, $\lambda = g_{YM}^2 N$, is large [3, 4][3]. Although this regime is a concrete realization of the Yang-Mills/String duality, the string theory side is somewhat crippled: only the lowest, massless string states contribute, and with few exceptions only tree level interactions have been investigated.

To study the relation between *quantized strings* and gauge theory in the AdS/CFT setting, one has to consider intermediate or small λ, and the limit of vanishing λ naturally presents itself as a manageable alternative zeroth order approximation. Then the string tension $T \sim \sqrt{\lambda}/R^2$ effectively goes to zero, if the radius of curvature R of the background is kept fixed. Or the radius of curvature becomes much smaller than the string length scale $l_s \sim T^{-1/2}$, i.e. $R/l_s \ll 1$. There are arguments [6] to all orders in the string coupling $g_s = g_{YM}^2$ and $\alpha' = l_s^2$ that the $AdS_5 \times S^5$ background is a solution to string theory, and it seems natural to assume that $\lambda = 0$ gauge theory is dual to (or can serve as a definition of) zero tension string theory on this background. Certainly, the two theories should both be symmetric under $SU(2, 2|4)$, acting as a superconformal group on the gauge theory, and as anti-deSitter supersymmetries on the string theory. Here we note that the problem of defining quantized tensionless or null strings in flat backgrounds [7] is in fact a more complicated problem than the present AdS case, due to its lack of a curvature scale, and its solution relies on additional assumptions.

Because the full $AdS_5 \times S^5$ background also contains a Ramond-Ramond field which prohibits the use of conventional string quantization methods, the quantization is a very difficult problem. Although interesting progress has been made [8, 9, 10, 11], we propose a different route. We assume that the strong version of Maldacena's conjecture works, i.e. that the gauge theory describes string theory even at small 't Hooft coupling. Then we can ask whether the picture of string theory that emerges is consistent with general expectations about the behaviour of string theory. Thus, one can get indirect evidence for or against the strong form of the AdS/CFT correspondence, by collecting knowledge about the gauge theory, which can be interpreted as knowledge about string theory until evidence is found to the contrary. Since we have not found any such negative evidence we will use gauge theory and string theory terminology interchangeably, but it should be remembered that all our calculations are done in gauge theory.

[3]For further references to these developments see the comprehensive review [5].

String theory can usually be characterized by its asymptotic states and interactions between them encoded in the scattering matrix. In an AdS background one immediately runs into conceptual problems, since neither the notion of asymptotic states nor of an ordinary S-matrix are well defined. Still, in terms of perturbations on the boundary of AdS, Balasubramanian et. al. [12] and Giddings [13] have argued for a kind of generalized S-matrix, which replaces the usual S-matrix for string theory in this background. It is also directly related to CFT correlation functions by the AdS/CFT correspondence.

While we cannot isolate ordinary asymptotic states in an AdS background, we can do equally well, at least in principle. The spectrum (of energy in global coordinates) is discrete, and we could study how interactions affect the states of the theory. In the zero λ limit we are considering, this is a purely combinatorial problem. The leading three-point functions of the gauge-invariant states which admit a string interpretation are of order $1/N \sim \kappa/R^4$, where κ is the gravitational coupling. To leading order in large N single-string states can be viewed as covariant strings of super-Yang-Mills string bits[4]. These AdS states correspond to CFT states, and by the CFT operator-state correspondence we could find associated operators, which are the operators involved in the generalized S-matrix.

For each AdS state there is a deformation of the string theory background [3, 4, 5]. The most important deformations are the relevant and marginal deformations, which do not ruin the UV properties of the CFT, or the asymptotically locally AdS nature of the corresponding spacetime. In section 2 we list all such (primary) operators composed exclusively of scalars. Surprisingly, we find several operators corresponding to massless spin two fields in the bulk. We also discuss how string states mix by $1/N$ corrections, and how the string propagator can be diagonalized.

In string theory, all the essential information about interactions is encoded in the three-string vertex. Similarly, the interactions in the conformal field theory are summarized in the operator product expansion. Not surprisingly three-string vertices and the OPE correspond closely to one another in the AdS/CFT dictionary. In section 3 we study general features like selection rules in the $\lambda = 0$ case, to leading order in large N, and also discuss some important special cases. We also dispel the fear that free field theory is too trivial to describe a complicated interacting string theory.

Given a generalized S-matrix we may discuss the properties of amplitudes. Relativistic amplitudes should obey crossing symmetry, whether they are point-particle amplitudes or string amplitudes, but whereas point-particle amplitudes can be obtained from sums of different Feynman diagrams with singularities in distinct crossed channels, string amplitudes come from string diagrams which by analytic continuation each exhibit singularities in several crossed channels. This property of string amplitudes was called "duality" in the early days of string theory. In section 4 we check the crossing symmetry of a particular CFT four-point function, which trans-

[4]String bits have been proposed by Thorn [14] as possible constituents of strings in a non-covariant formulation.

lates to duality of the generalized string four-point amplitude. We also indicate a simple direct argument for general crossing symmetry in the kind of CFT built on free field theory that we are considering.

2 States and propagators

In addition to the gauge potential the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory contains six scalars in the adjoint representation of the gauge group, as well as fermions. Local conformal operators may be written as products of fundamental fields in the adjoint representation (the field strength in the case of the gauge potential). Covariant derivatives (ordinary derivatives at $\lambda = 0$) on the fundamental fields are also allowed. When the trace of the product is taken one gets invariants. Because of the cyclic symmetry of the trace, we may think of the single-trace operators as necklaces (closed strings) composed of SYM beads (string bits). Multiple-trace operators, i.e. products of single-trace operators, correspond to multi-string states. In [15] the full spectrum of single-trace fields in the zero λ limit is given, but in this paper we instead focus on some general features of correlation functions/string amplitudes. At zero λ different fundamental fields propagate independently so it is perfectly consistent to restrict attention to a subset of them. For simplicity we only consider conformal operators built of the six scalar fields ϕ^I:

$$
\begin{aligned}
(\partial^{\{n\}}\Phi^{\{I\}})_{\{\mu\}} &\equiv (\partial^{n_1\ldots n_k}\Phi^{I_1\ldots I_k})_{\mu_1^1\ldots\mu_1^{n_1}\ldots\mu_k^1\ldots\mu_k^{n_k}} \\
&\equiv \frac{1}{N^{k/2}}\mathrm{Tr}\left\{(\partial_{\mu_1^1}\ldots\partial_{\mu_1^{n_1}}\phi^{I_1})\ldots(\partial_{\mu_k^1}\ldots\partial_{\mu_k^{n_k}}\phi^{I_k})\right\}, \qquad (1)
\end{aligned}
$$

where we have introduced multiple indices denoted with braces. Note that Hermitean operators generally are special linear combinations of such operators.

We study operators of definite conformal dimension. In our simple setting without interactions, the dimension is additive. The fundamental scalar has dimension $\Delta_\phi = 1$ and the derivative (the translation generator) has $\Delta_\partial = 1$. Primary operators are operators which (at the origin) are annihilated by special conformal transformations. From them descendant operators, said to belong to the same conformal family, are created by repeated application of the other conformal generators, in effect the derivative. In the AdS picture the primary operator gives a ground state for the Hamiltonian conjugate to the global time coordinate, and the descendants are excited states, which may be obtained by acting with AdS isometries not commuting with the Hamiltonian. Thus all the particles in AdS can be listed by only listing the corresponding conformal primaries. It is also enough to consider the correlation functions of the primaries, since those of descendants are related by conformal symmetry.

The propagator of a scalar field in the adjoint representation of $SU(N)$ is

$$
\langle\phi^\alpha_\beta(x)\phi^\gamma_\delta(y)\rangle = (\delta^\alpha_\delta\delta^\gamma_\beta - \frac{1}{N}\delta^\alpha_\beta\delta^\gamma_\delta)|x-y|^{-2\Delta_\phi} \qquad (2)
$$

where the first term is the only one for the group $U(N)$, allowing for 't Hooft's double line representation [1] in the large N limit. For $SU(N)$ the second term can be dealt with by $1/N$ corrections to the naive double line diagrams. The above propagator for fundamental scalars can be used to calculate any correlation function

$$\langle\, \partial^{\{n_1\}}\Phi^{\{I_1\}}(x_1)\ \ \partial^{\{n_2\}}\Phi^{\{I_2\}}(x_2)\ldots\ \partial^{\{n_m\}}\Phi^{\{I_m\}}(x_m)\,\rangle \tag{3}$$

in the $\lambda = 0$ limit, e.g. by making all possible contractions directly, or by using Wick's theorem (all conformal operators are defined to be normal ordered). In particular any scalar two point function may be calculated, and the results give a metric in the space of operators,

$$\langle A(x_i)B(x_j)\rangle \equiv G_{AB}|x_{ij}|^{-\Delta_A-\Delta_B} \equiv \mathcal{G}_{AB}, \tag{4}$$

where we have defined

$$x_{ij} \equiv x_i - x_j. \tag{5}$$

Two-point functions of non-scalars scale in the same way but G_{AB} then depends on polarizations and the direction of x^μ_{ij}. The conformal operators in eq. 1 have been normalized to have leading N-independent terms in large N two-point functions (with their Hermitean conjugates).

The value of the two point correlator (modulo its spacetime dependence) of a primary operator with the Hermitean conjugate of another operator works as an inner product[5] in the space of primaries. The same quantity for any operators we call "overlap", by abuse of terminology. Descendants of two orthogonal primaries have vanishing overlap with one another. Conversely, a vanishing two point function between two operators means that they belong to orthogonal conformal families. Therefore, an operator is primary if and only if it has vanishing overlap with all operators of lower dimension.

All operators free of derivatives are primary, simply because there are no operators with lower dimensions that can have non-zero overlap with them. But there are also numerous primaries containing derivatives, the most commonly known being conserved $SO(6)$ currents and the conserved stress tensor. For free scalars

$$J^{IJ}_\mu \;=\; \frac{1}{N}\mathrm{Tr}\left\{\phi^I\partial_\mu\phi^J - \phi^J\partial_\mu\phi^I\right\} = \partial^{01}\Phi^{IJ}_\mu - \partial^{01}\Phi^{JI}_\mu \tag{6}$$

$$T_{\mu\nu} \;=\; \frac{\mathrm{const}}{N}\mathrm{Tr}\left\{\partial_\mu\phi^I\partial_\nu\phi^I - \frac{\eta_{\mu\nu}}{4}\partial_\rho\phi^I\partial^\rho\phi^I - \frac{1}{2}\phi^I\partial_\mu\partial_\nu\phi^I + \frac{\eta_{\mu\nu}}{8}\phi^I\partial^2\phi^I\right\} \tag{7}$$

In our case it is also easy to construct other primaries which are linear combinations of terms with a single derivative. The operators

$$\mathrm{Tr}(\phi^{I_1}\ldots\partial_\mu\phi^{I_k}\ldots\phi^{I_l}\cdots\phi^{I_m}) - \mathrm{Tr}(\phi^{I_1}\ldots\phi^{I_k}\ldots\partial_\mu\phi^{I_l}\ldots\phi^{I_m}) \tag{8}$$

[5] For Hermitean operators it is just a component of the metric g_{AB}, as seen in eq. 4.

	$\Delta = 2$	$\Delta = 3$	$\Delta = 4$
Φ^{IJ}	$\ydiagram{2} \oplus \bullet$		
Φ^{IJK}		$\ydiagram{3} \oplus \ydiagram{1} \oplus \ydiagram{1,1,1}$	
Φ^{IJKL}			$\ydiagram{4} \oplus 2\,\ydiagram{2} \oplus 2\bullet \oplus 2\,\ydiagram{2,2} \oplus 2\,\ydiagram{2,1,1} \oplus \ydiagram{1,1}$
$(\partial^{10}\Phi^{IJ})_\mu$		$\ydiagram{1,1}$	
$(\partial^{100}\Phi^{IJK})_\mu$			$2\,\ydiagram{2,1} \oplus 2\,\ydiagram{1}$
$(\partial^{\{\Sigma n=2\}}\Phi^{IJ})_{\mu\nu}$			$\ydiagram{2} \oplus \bullet$

Table 1: Spectrum of relevant and marginal primaries.

can only have non-zero overlap with operators composed of the same fields. Up to permutations of the fundamental fields the only such operators of lower dimension are

$$\mathrm{Tr}(\phi^{I_1} \dots \phi^{I_k} \dots \phi^{I_l} \dots \phi^{I_m}), \tag{9}$$

which by construction have vanishing overlap with the operators (8). Therefore, expression (8) represents a new primary unless it vanishes, which it does if the derivatives happen to act on identical fields in cylically equivalent positions. There are also many primaries with more than one derivative, but such operators are more difficult to generate.

The most important operators are the IR relevant and marginal operators, which can be added to the Lagrangian without destroying the UV behaviour. They have $\Delta \leq 4$ and are given in table 1 in terms of their composition, derivative structure and $SO(6)$ Young tableaux. The table was constructed by checking the effect of the cyclic property of the trace, which projects out some operators and relates others. The primaries were then picked out. There are more marginal and relevant gauge invariant primaries composed solely of scalars in the $\lambda = 0$ limit[6] than in the supergravity limit $\lambda \to \infty$. In the supergravity limit the only such scalar primaries are symmetric traceless tensors of $SO(6)$ [4, 5]. This indicates an intricate structure of branches for the moduli space of the theory, with new branches of conformal field theory splitting off at intermediate values of λ, where some operators relevant at

[6] In the canonical normalization of fundamental fields of eq. 2, a marginal perturbation to a non-zero λ theory can only be achieved by adding interaction terms to the Lagrangian which break the original Abelian gauge symmetries and replace them with a deformed, non-Abelian gauge symmetry.

$\lambda = 0$ become marginal. At least we could expect that the possible IR limits of deformations of the theory vary strongly with the UV coupling λ. Another surprise in table 1 is the last line, with 20 $SO(6)$ traceless symmetric tensors, which are symmetric traceless in spacetime, as well as conserved. In AdS they are $SO(6)$ charged massless spin two cousins of the graviton! If we had taken vector fields into account we would also have listed the vector contribution to the energy momentum tensor, which is an $SO(6)$ scalar, and corresponds to a second AdS field with the quantum numbers of the graviton. At this time it is too early to say whether these curious facts imply that there is something seriously wrong with the zero coupling theory, or if they have something profound to tell us about stringy geometry.

Even if one has chosen a basis of mutually orthogonal primaries in the large N limit, there will in general be $1/N$ corrections to two-point functions which mix originally independent operators. This is the most basic way in which a kind of interactions appear in our free theory, and it is a string coupling of the order of $1/N$ at work. A few examples computed in the $U(N)$ theory illustrates how the general computation consists of a combinatorial part and an analytic part, which takes care of the polarization dependence of the two-point function.

$$\begin{aligned}\left\langle \Phi^{123}(x)\Phi^{123}(0)\right\rangle &= \frac{1}{N^3} <: \left[\mathrm{Tr}(\phi^1\phi^2\phi^3)\right](x) :: \left[\mathrm{Tr}(\phi^1\phi^2\phi^3)\right](0) :> \\ &= \frac{1}{N^2}|x|^{-6}\end{aligned} \tag{10}$$

$$\begin{aligned}&\left\langle \left[\Phi^{12}\Phi^{13}\right](x)\ \Phi^{1231}(0)\right\rangle \\ &= \frac{1}{N^4}\left\langle : \left[\mathrm{Tr}(\phi^1\phi^2)\mathrm{Tr}(\phi^1\phi^3)\right](x) :: \left[\mathrm{Tr}(\phi^1\phi^2\phi^1\phi^3)\right](0) :\right\rangle \\ &= \frac{1}{N}|x|^{-8} + \frac{1}{N^3}|x|^{-8}\end{aligned} \tag{11}$$

$$\begin{aligned}&\left\langle J^{12}_\mu(x)J^{12}_\nu(y)\right\rangle \\ &= \frac{1}{N^2}\left\langle : \left[\mathrm{Tr}(\phi^1\partial_\mu\phi^2 - \phi^2\partial_\mu\phi^1)\right](x) :: \left[\mathrm{Tr}(\phi^1\partial_\nu\phi^2 - \phi^2\partial_\nu\phi^1)\right](y) :\right\rangle \\ &= 2|x-y|^{-2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}|x-y|^{-2} - 2\frac{\partial}{\partial y^\nu}|x-y|^{-2}\frac{\partial}{\partial x^\mu}|x-y|^{-2}\end{aligned} \tag{12}$$

The combinatorial calculation involves counting how many closed index lines are formed between the two operators in the double line representation, to give the appropriate N-dependence. Note that the convention of normal ordering operators just means that propagators should not return to the same operator. If there are several ways of saturating the operators with propagators, they should be added, and will in general give rise to a polynomial dependence on $1/N$.

To give the two-point function a simple and physical form, one should diagonalize the mixing matrix. Because primary operators can only mix with operators of the

same dimension, and two operators that mix also have to consist of equal numbers of all fundamental fields, the mixing problem can be reduced to a block diagonal form. Only finite-dimensional diagonalizations are needed to find the exact propagator.

Single-trace operators may also mix with multiple-trace operators, i.e. products of single-trace operators. A natural AdS interpretation of such operators is as multi-string states, but it is somewhat puzzling that such products of independent operators with equal argument should play a special role, in addition to being limits of products of unequal arguments. Presumably the normal ordering needed to regularize the product can be interpreted in AdS as a way of binding the two strings to each other in the radial direction (which in the AdS correspondence is related to the boundary theory scale [16, 17]).

If one includes the multiple-trace operators among the operators that can mix, one gets larger matrices to diagonalize, but still of finite dimension, by the same argument as before. The resultant diagonalized full propagator propagates N-dependent linear combinations of single-trace and multi-trace operators, without mixing among these superpositions. Their dimensions are all unchanged, and N-independent. This result about N-independence at $\lambda = 0$ sharpens the assertion in [18] about the behaviour of the dimensions of multi-trace operators at weak coupling. In contrast, the *strong* 't Hooft coupling result of D'Hoker et al [18] is that the dimensions of multi-trace operators do shift.

The block overlap matrices should become degenerate for some finite N, depending on the block. This is because there are linear dependencies among the naive states [15], known as a string exclusion principle [19]. The determinants of the block overlap matrices are polynomials in $1/N$, so the smallest root of each determinant sets the value of N for which $1/N$ perturbation theory breaks down in the given block.

3 Operator products and string vertices

Essentially all string theory interactions may be derived from three-string vertices, roughly because all string diagrams can be constructed by sewing together pant diagrams (which carry the three-string structure). In many approaches additional contact terms are also needed, but their role is mainly to make sense of analytic continuations. Similarly, in conformal field theory we expect the three-point functions (and analytic continuation) to be enough to calculate any correlation function. The three-point functions contain essentially the same information as the operator product expansion, which completely characterizes the theory if conformal bootstrap [20] works as in two dimensions [21]. For general operators the three-point function

$$\begin{aligned}&\langle A(x_1)B(x_2)C(x_3)\rangle\\ &\quad = \frac{C_{ABC}}{|x_{12}|^{\Delta_A+\Delta_B-\Delta_C}|x_{31}|^{\Delta_A+\Delta_C-\Delta_B}|x_{23}|^{\Delta_C+\Delta_B-\Delta_A}} \equiv \mathcal{C}_{ABC}, \qquad (13)\end{aligned}$$

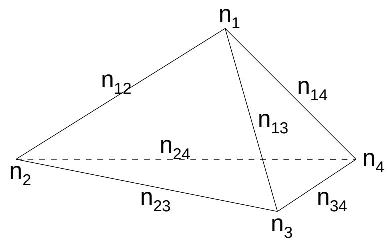

Figure 1: A schematic description of a four-point function as a tetrahedron. To resolve the N-dependence the order of the fields in each operator also has to be taken into account.

where spacetime dependence is included in $\mathcal{C}_{ABC}$, which as C_{ABC} typically depends on the spins of the operators and the relative orientations of the x^{μ}_{ij}. The general operator product expansion

$$A(x)B(y) \sim \sum_{D} C_{AB}{}^{D} D(y)|x-y|^{\Delta_D-\Delta_A-\Delta_B} = \sum_{\mathcal{D}} \mathcal{C}_{AB}{}^{D} D(y) \tag{14}$$

is formally related to the three-point function through

$$\mathcal{C}_{AB}{}^{D} \equiv \mathcal{C}_{ABC}\, \mathcal{G}^{CD}, \tag{15}$$

with $\mathcal{G}^{AB}$ the inverse of the propagator $\mathcal{G}_{AB}$.

The n-point functions are constrained by the requirement that all fundamental fields should be joined by a propagator to a fundamental field in another operator (see fig. 1). This implies that all non-zero correlation functions contain an even number of fundamental fields. Furthermore, any n-point function can be represented by an n-hedron for each kind of fundamental field (see fig. 1). There are n^I_i fields ϕ^I at the i-th corner, and n^I_{ij} propagators of ϕ^I along the $ij \equiv ji$ edge. We must have $n^I_i = n^I_{i1} + \ldots + n^I_{in}$ and

$$n^I_{ij} = \frac{1}{n-2}(n^I_i + n^I_j - \frac{n^I_1 + \ldots + n^I_n}{n-1}) \tag{16}$$

For the three-point function, $n = 3$, a non-negative number of propagators $n^I_{ij} \geq 0$ implies triangle inequalities

$$n^I_{\pi(1)} \leq n^I_{\pi(2)} + n^I_{\pi(3)} \tag{17}$$

for any permutation π.

The underlying reason for the rules above is that we are dealing with a free theory, which is invariant to independent shifts of all the fundamental scalars. The corresponding conserved currents are $J^{I\alpha}_{\mu\beta} = \partial_\mu \phi^{I\alpha}_{\beta}$, which are not gauge singlets, and thus not among the operators we would otherwise consider.

In our case we have a free theory, and the OPE can be obtained by first applying Wick's theorem and then Taylor expanding the result. For example we have

$$\begin{aligned}
&\frac{1}{N} :\mathrm{Tr}\{\phi^2(x_i)\}: \ \frac{1}{N} :\mathrm{Tr}\{\phi^2(x_j)\}: = \\
&\quad = \frac{1}{N^2} : \mathrm{Tr}\{\phi^2(x_i)\}\mathrm{Tr}\{\phi^2(x_j)\} : + \frac{4}{N^2|x_{ij}|^2} : \mathrm{Tr}\{\phi(x_i)\phi(x_j)\} : + \frac{4}{|x_{ij}|^4} \\
&\quad = \frac{1}{N^2} \sum_n \frac{(x_{ij})^{\mu_1} \dots (x_{ij})^{\mu_n}}{n!} : \mathrm{Tr}\{\partial_{\mu_1} \dots \partial_{\mu_n} \phi^2(x_j)\}\mathrm{Tr}\{\phi^2(x_j)\} : \\
&\quad + \frac{4}{N^2|x_{ij}|^2} \sum_n \frac{(x_{ij})^{\mu_1} \dots (x_{ij})^{\mu_n}}{n!} : \mathrm{Tr}\{\partial_{\mu_1} \dots \partial_{\mu_n} \phi(x_j)\phi(x_j)\} : + \frac{4}{|x_{ij}|^4}. \qquad (18)
\end{aligned}$$

Terms proportional to the unit operator do not contribute to three-point functions, but as we will see explicitly in section 4, they are essential for the $1/N$-expansion to produce disconnected diagrams. Such diagrams are of course needed if the expansion is to be interpreted as a perturbative expansion of string theory.

Since the model is essentially a trivial free field theory, only studied from the special perspective of its gauge-invariant local operators, we might worry that the corresponding string theory is also trivial. In particular, we might ask if there are only lowest order, $1/N$, string interactions. Could it be that diagonalization of the full two-point function is enough, and absorbs all other N-dependence? For flat space amplitudes, such behaviour would be impossible in an interacting theory because of S-matrix unitarity[7]. In the present theory, we do not have an ordinary S-matrix, neither in the four-dimensional Minkowski space because of conformal invariance, nor in the five-dimensional gravitational picture because of the AdS background, so this argument does not necessarily apply. To resolve the issue we have found a three-point function with only higher order interactions, and checked that diagonalization of the two-point functions of the three operators cannot reduce the interaction to order $1/N$, the coupling strength of the fundamental interactions. This demonstrates that the theory is a highly non-trivial interacting theory even at zero 't Hooft coupling (i.e. for tensionless strings).

Consider the correlation function

$$\langle \Phi^{1212}(x_1)\Phi^{2323}(x_2)\Phi^{3131}(x_3)\rangle \qquad (19)$$

among single-trace operators. The leading contributions are shown in fig. 2, and they are of order $1/N^3$. The three operators involve different fields and cannot mix pairwise with each other. Thus, no diagonalization of single-trace operators can give this three-point function from a $1/N$ vertex and $1/N$-corrected external states.

By diagonalization among the full set of gauge invariant operators, including multi-trace operators, it is possible to get terms like the leading contribution as a result of an admixture of double-trace operator in the external state, but again

[7] It generates an order g^2 imaginary part from an interaction of order g, etc...

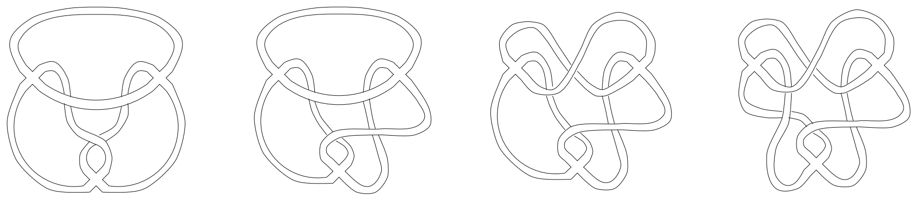

Figure 2: Leading diagrams contributing to the correlator (19).

vertices of higher order than $1/N$ are needed to couple to the remaining two states. To see this factorize a diagram of fig. 2 into a first factor consisting of $1/N^2$ three-point vertices $\langle \Phi^{IJIJ}\Phi^{JKJK}[\partial^{\{n\}}\Phi^{KI}\partial^{\{m\}}\Phi^{KI}]\rangle$ between two external single-trace operators and two-trace operators and a second factor consisting of a $1/N$ mixing $\langle [\partial^{\{n\}}\Phi^{KI}\partial^{\{m\}}\Phi^{KI}]\Phi^{KIKI}\rangle$ of such two-trace operators with the remaining external single-trace operator. This example indicates that it may be possible to write the full theory in terms of completely diagonalized local operators, but also that three-point vertices of higher order in $1/N$ are needed.

4 CFT crossing symmetry and string amplitude duality

The OPE, eq. 14, can be used inside correlation functions in several ways depending on which distances are assumed to be small, and at what points the operator products are to be inserted. By a sequence of expansions an n-point function can be reduced to operator product coefficients ${\mathcal{C}_{AB}}^D$ joined by operator two-point functions $\langle C(x_i)D(x_j)\rangle$, all multiplied together and summed over all possible propagating operators. More symmetrically, the n-point function may be expressed in terms of two-point functions and the amputated three-point function $\mathcal{C}^{ABC}$, obtained by multiplying the three-point function by inverse two-point functions. For example a six-point function can be written as

$$\begin{aligned}&{\mathcal{C}_{AB}}^{B'}{\mathcal{C}_{DC}}^{C'}{\mathcal{C}_{FE}}^{E'}{\mathcal{C}_{B'C'}}^{C''}\mathcal{G}_{C''E'}\\ &= \mathcal{G}_{AA_1}\mathcal{G}_{BB_1}\mathcal{G}_{CC_1}\mathcal{G}_{DD_1}\mathcal{G}_{EE_1}\mathcal{G}_{FF_1}\mathcal{C}^{A_1B_1B'}\mathcal{C}^{C_1D_1C'}\mathcal{C}^{E_1F_1E'}\mathcal{G}_{B'B'_1}\mathcal{G}_{C'C'_1}\mathcal{G}_{E'E'_1}\mathcal{C}^{B'_1C'_1E'_1}\end{aligned} \tag{20}$$

Diagrammatically this may be drawn as in fig. 3. In the second way of writing the six-point function additional internal spacetime points serving as arguments of two-point functions and amputated three-point functions are introduced. If only the expansions converge the locations of these points are arbitrary.

No single sequence of expansions converges for all positions of the operators, but one can hope that different sequences should converge in complementary regions in the space of operator positions, and yield continuations of each other. Interpreted in terms of an AdS string S-matrix this would mean that the amplitude could be

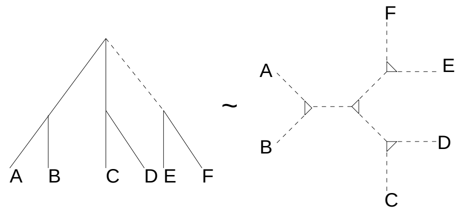

Figure 3: Diagrams showing two possible expansions of a six-point functoion given in eq. 20.

expanded in kinematic invariants in many different ways that are continuations of one another. But this is just the kind of scattering "duality" that was the origin of string theory [22], and which is intuitively reasonable when the amplitude is viewed as a result of a path integral over string world sheets with no interaction points (in contrast to corresponding amplitudes for point-particle theory). In our case we do not have an actual world-sheet picture, just sets of free particle propagators that can span a surface thanks to large N counting. But by examining a four-point function we can see explicitly how string scattering duality emerges from the OPE of conformal field theory.

If we insert the OPE eq. 18 twice into the four-point function of normalized quadratic traces of $U(N)$ scalars

$$\frac{1}{N^4}\left\langle :\mathrm{Tr}\,\phi^2(x_1): :\mathrm{Tr}\,\phi^2(x_2): :\mathrm{Tr}\,\phi^2(x_3): :\mathrm{Tr}\,\phi^2(x_4):\right\rangle \tag{21}$$

we get

$$\begin{aligned}
&\frac{1}{N^4}\left\langle :\mathrm{Tr}\,\phi^2(x_1): :\mathrm{Tr}\,\phi^2(x_2): :\mathrm{Tr}\,\phi^2(x_3): :\mathrm{Tr}\,\phi^2(x_4):\right\rangle \qquad (22)\\
&= \frac{16}{|x_{12}|^4|x_{34}|^4} + \frac{1}{N^4}\sum_{n,m}\frac{(x_{12})^{\mu_1}\ldots(x_{12})^{\mu_n}}{n!}\frac{(x_{34})^{\nu_1}\ldots(x_{34})^{\nu_m}}{m!}\times\\
&\quad\times\Big(\Big\langle :[\mathrm{Tr}\,\partial_{\mu_1}\ldots\partial_{\mu_n}\phi^2\ \mathrm{Tr}\,\phi^2](x_2): :\Big[\mathrm{Tr}\,\partial_{\nu_1}\ldots\partial_{\nu_m}\phi^2\ \mathrm{Tr}\,\phi^2\Big](x_4): \Big\rangle\\
&\qquad + \frac{16}{|x_{12}|^2|x_{34}|^2}\left\langle :[\mathrm{Tr}\,\phi\,\partial_{\mu_1}\ldots\partial_{\mu_n}\phi]\,(x_2): :[\mathrm{Tr}\,\phi\,\partial_{\nu_1}\ldots\partial_{\nu_m}\phi]\,(x_4): \right\rangle\Big),
\end{aligned}$$

for small x_{12} and x_{34} relative to x_{23} and x_{14}. The first term comes from the terms proportional to the unit operator in the OPEs, corresponds to disconnected diagrams. The second line on the right-hand side corresponds to the propagation of quartic operators, and consists of one connected and two disconnected pieces (corresponding to propagation of double-trace operators). Finally, the last line consists of connected diagrams propagating quadratic operators. A direct Taylor expansion of the Green function (21) gives the same result as this double OPE, and the regions of convergence are the same. There are three different ways of combining the four

external operators into two pairs, each yielding a different expansion of the same Green function. Therefore, the full Green function can be obtained as a continuation of expansions in any such channel. This is string scattering duality for the corresponding AdS amplitude.

The basic reason for the above duality appears to be that products of normal ordered operators are associative. Presumably the associativity can be used to prove rigorously many of the formal identities discussed above relating n-point functions, OPE coefficients, three-point functions and two-point functions.

5 Discussion

We have used the AdS/CFT conjecture as a tool to tentatively define string theory in $AdS_5 \times S^5$ with a Ramond-Ramond background. Although we have used the correspondence outside the region where it has been tested, at small 't Hooft coupling, we have found that such a definition gives rise to a non-trivial interacting theory with the fundamental properties of a string theory, like duality of scattering amplitudes. We have tested a simple four-point amplitude and verified that CFT crossing symmetry gives rise to the desired behaviour. We have listed marginal and relevant primary operators composed of scalars and found that there are more such operators at small 't Hooft coupling than at large, indicating a complicated phase diagram of IR deformations of $\mathcal{N} = 4$ SYM. In string theory we expect a large number of backgrounds which are asymptotically AdS.

Surprisingly we have found several marginal traceless symmetric tensors, which correspond to massless spin 2 particles in AdS. Somehow, the extremely stringy tensionless limit involves several geometries interacting with each other. It remains to be seen if this is a defect which can only be cured by a perturbation to non-zero tension, or if it is a consistent and perhaps even a characteristic property of string theory at extremely short distances.

Furthermore we have argued that the theory in the limit of vanishing 't Hooft coupling allows a complete diagonalization of the string propagator. Nevertheless, we have found the theory to be a complicated interacting theory with interactions of all orders in $1/N$.

A puzzling question is if the purely combinatorial $1/N$ expansion in the zero coupling theory, which by large N lore is the genus expansion of string theory, can be related to sums of intermediate single- and multiple-string states. In particular, one would expect string loops to correspond to integrals or sums over *all* multi-string intermediate states (composed of arbitrarily many fundamental fields) that can couple to the external states. For loop sums to equal the combinatorial sums there apparently have to be enormous cancellations, since the number of fundamental propagators in the sums is bounded by expressions like eq. 16. Perhaps such cancellations are typical of extremely holographic systems.

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